FINDING YOUR WAY AROUND THE ARCTIC USING GPS

The GPS satellites that orbit the Earth continually transmit messages containing the exact time of transmission, and the exact location of the satellite at the time of transmission. When a GPS receiver receives these messages, it uses the time delay to work out exactly how far away it is from the satellite.

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Ann checking her GPS position before setting off in the morning

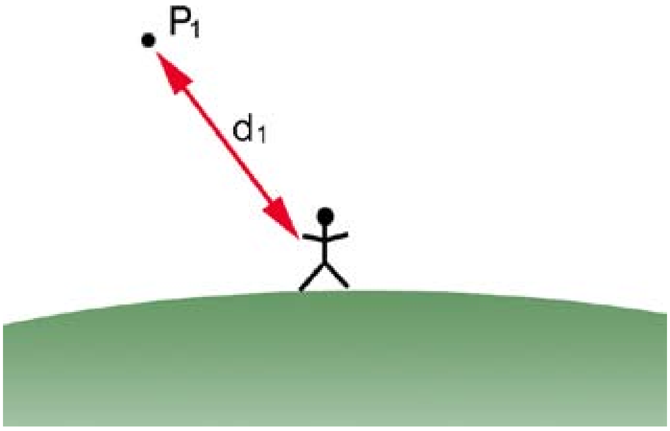
Question 1: The signal transmitted by a satellite travels at the speed of light, which is approximately 300,000 km/s. The ice team's GPS receiver receives a signal from a satellite and calculates that the time the signal took to reach the receiver was 0.07 seconds. How far is the receiver from the satellite?

* ***Answer:*** *Write d for the distance between satellite and receiver in kilometres. Then d/300,000 = 0.07, so d = 0.07 x 300,000 = 21,000, so the satellite is 21,000 km away from Earth.*

“Without GPS technology, the ice team would have to navigate by the stars!”

Simon Harris-Ward
Operations
Director

Question 2: Suppose the receiver has calculated that it is a distance d_1 away from a particular satellite (for simplicity, imagine that the satellite is stationary at point P_1). In the plane containing the satellite and the receiver, what kind of shape is formed by all the points that lie at distance d_1 from the satellite?



A two-dimensional view

Answer: *A circle with radius* d *and centre* P_1.

Question 3: Is one satellite enough to calculate the explorers' exact location?

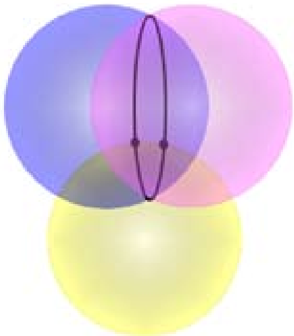
Answer: *No, if the satellite is not directly overhead the circle of points at distance* d_1 *would intersect the Earth's surface at two points, and either of those could be the location of the explorers. It would be possible to determine their exact location if the satellite is directly overhead, but there is no way to know if this is the case.*

Question 4a: **In a two-dimensional coordinate system mark the points P_1 and P_2 with coordinates (0,20) and (30,20) respectively. These denote the locations of two satellites that lie in a plane with the explorers (one unit in the coordinate system corresponds to 1000 km in our two-dimensional world, ground level lies on the x-axis). The receiver has calculated that the explorers are exactly 25 units (25,000km) away from each of the two satellites. Can you find the exact location of the explorers by drawing circles centred on the two satellites?**

Question 4b (harder): **In a two-dimensional coordinate system mark the P_1 and P_2 with coordinates (0,20) and (30,20) respectively. These denote the locations of two satellites that lie in a plane with the explorers (one unit in the coordinate system corresponds to 1000 km in our two-dimensonal world, ground level lies on the x-axis). The receiver has calculated that the explorers are exactly 25 units (25,000km) away from each of the two satellites. Can you calculate the exact location of the explorers by using the equations of the circles centred on the two satellites?**

* ***Answer:*** *Using a compass to draw two circles around the two points gives coordinates (15,0) and (15,40), hence the explorers are at (15,0) as they are at ground level. Same result if you use the circle equations* $x^2+(y-20)^2=25^2$ *and* $(x-30)^2+(y-20)^2=25^2$*.*

When the satellites and explorers are all in a plane, the data from two satellites is enough to calculate the exact position of the explorers. Any two circles in the plane meet in one point, two points or not at all. Since the explorers are on each of the circles centred on the satellites, the two circles must meet. If they meet at just one point we are done. If they meet at two points, then we use the fact that we know which side of the satellites the explorers are (they are on Earth) to work out which of the two intersection points gives the location.



Two intersecting spheres generally meet in a circle and a third meets this circle in at most two points.

Question 5: Now consider the more general case, when the satellites and the explorers do not all lie in a plane. In three dimensions, what kind of shape is formed by all the points at distance d from the satellite at point P? Are two satellites enough to calculate the exact location of the explorers in three dimensions? How do questions 2 and 3 generalise to three dimensions?

* ***Answer:*** *In three dimensions, the points at distance d from a given point P lie on a sphere with centre P and radius d. The points that lie at distance d_1 from point P_1 and at distance d_2 from P_2 lie on a circle C formed by the intersection of the two spheres centred at P_1 and P_2 with radii d_1 and d_2 respectively. So we need at least one other satellite to determine the explorers' position. One other satellite is sufficient because a third sphere will intersect the circle C in at most two points. Additional information will then allow you to work out which of these is the location of the explorers.*

Question 6a (challenging): Write down the equations of the sphere centred at the point (0,15,0) with radius 25, and the sphere centred at (0,-15,0) with radius 25. Observing the symmetry, what can you say about the circle of intersection?

* ***Answer:*** *Sphere 1: $x^2+(y-15)^2+z^2=25^2$*
Sphere 2: $x^2+(y+15)^2+z^2=25^2$.
As the centres lie 15 units either side of the origin along the y axis, and since the spheres have the same radius, the circle of intersection must lie in the (x,z) plane and be centred at (0,0,0).

Question 6b (challenging): What are the centre and radius of the circle C of intersection?

* ***Answer:*** *It's the circle centred at (0,0,0) with radius 20*

Question 6c (challenging): Observing the relative positions of the circle C of intersection and the sphere centred at the point (0,0,40) with radius 20, find the point(s) where the sphere and the circle meet.

* ***Answer:*** *The centres of the circle and the sphere both lie on the z-axis and are 40 units apart. Since the radii of the sphere and the circle are both 20, there is only one point of intersection, namely (0,0,20).*

Conclusion: GPS systems are based on this process, which is called *trilateration*. However, to minimise errors and to gain extra information, real GPS systems calculate the distance between the receiver and four satellites.