

Navigation Toolkit

HOW TO FIND YOUR WAY AROUND THE ARCTIC

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A science and mathematics enrichment toolkit for ages 14-19 (Key Stages 4 and 5)

Produced by the Millennium Mathematics Project, University of Cambridge, for Arctic Survey Education

Navigation Toolkit – Contents

This enrichment toolkit explores some of the science that underlies the Catlin Arctic Survey and gives learners the chance to see curriculum science and maths applied to real-life problems. The overview article can be read on its own, or used as motivational material for the two worksheets. The worksheets are designed to promote group discussion of the topics, as well as provide hands-on activities. Because of the exploratory nature of the worksheets, less experienced students may require more guidance.

p4 Finding your way around the Arctic – background article

The aim of the Catlin Arctic Survey is to gather scientific information needed to understand the future of the Arctic sea ice cover. It's not just about heading for the North Pole, but about getting there along a route that enables the ice team to gather as much data as possible. But how do you find your way around the Arctic? This article gives background material on how mathematics is used to create maps of the Earth's surface, and how GPS is used to exactly locate the ice team's position. The article can either be read on its own, as enrichment material showing real world applications of trigonometry and geometry, or can provide motivational material for the two worksheets.

Mapping the Arctic with mathematics

p10 Mapping worksheet

p13 Mapping worksheet guidance with answers

This activity explores how the surface of the Earth is mapped onto a two-dimensional page. Students are encouraged to experiment with map projections using simple trigonometry, and investigate the limitations of different projections. The worksheet contains six questions, including some hands-on activities and some intended to stimulate an open discussion in the classroom.

Learning outcomes: Understand that there are different map projections, each with a certain amount of distortion. Use trigonometry and draw graphs of trigonometric functions.

Finding your way around the Arctic using GPS

p18 GPS worksheet

p21 GPS worksheet guidance with answers

This activity explains how to calculate exact locations using the information from GPS (global positioning system) satellites. Students use circles and spheres to describe points equidistant from satellite positions, and investigate how the intersection of these shapes can determine the location of the ice team. The worksheet contains six hands-on questions (including some challenging questions for more advanced or motivated students) that encourage the students to develop the GPS method and calculate exact locations for the ice team.

Learning outcomes: Understand how GPS works using 2D and 3D geometry. Use coordinate systems and equations of circles and spheres.

Arctic Survey Education – new facts and knowledge to promote understanding

Catlin Arctic Survey

The Catlin Arctic Survey is an international collaboration between polar explorers, some of the world's foremost scientific bodies and WWF. The intention is to better predict how much time there is before the North Pole sea ice cover melts by collecting and analysing new accurate data on snow and sea ice thickness gathered by the expedition team as they journey across the Arctic.

The scientific endeavour began on 1st March 2009. The expedition was led by highly experienced polar explorer and Expedition Leader, Pen Hadow. He was accompanied by Ann Daniels, one of the world's foremost female polar explorers and Martin Hartley, the leading expedition photographer. The team travelled on foot, having hauled sledges from 81°N 130°W across drifting sea ice, for 73 days, in temperatures from 0°C to -50°C towards the North Geographic Pole.

Current estimates for the total disappearance of the Arctic Ocean's sea ice vary from 50 years down to just four. Whatever happens, the consequences of its melt-down will be of global significance in terms of sea level rise, the geo-politics of energy resources, rainfall patterns and the availability of water supplies and, of course, the impact on biodiversity, including the polar bear.

Millennium Mathematics Project – bringing mathematics to life

The Millennium Mathematics Project (MMP) is a maths education initiative for ages 5 to 19 and the general public, based at the University of Cambridge but active nationally and internationally. We aim to support maths education and promote the development of mathematical skills and understanding, particularly through enrichment activities. More broadly, we want to help everyone share in the excitement and understand the importance of mathematics. The MMP consists of a family of complementary programmes, each with a different focus, including Plus Magazine (Plus).

This enrichment toolkit was produced by Plus, a free online magazine (<http://plus.maths.org>) for GCSE, A level and undergraduate students, and the general public. Plus opens a door to the world of maths, with all its beauty and applications, by providing articles from the top mathematicians and science writers on topics as diverse as art, medicine, cosmology and sport. Plus provides weekly news updates, a fortnightly email newsletter, an extensive 'Careers with Maths' library and monthly podcasts.

FINDING YOUR WAY AROUND THE ARCTIC

Where am I going?

The aim of the Catlin Arctic Survey is to gather information to better understand the future of the Arctic sea ice cover. It's not just about getting to the North Pole, but about getting there along a route that enables the ice team to gather as much data as possible. The route taken by the ice team has been carefully planned with the help of scientists using a detailed map of the Arctic – but how do you make such a map?

“Good navigation tools are essential – our life depends on them!”

Pen Hadow
Expedition Leader

photograph © Martin Hartley www.martinhartley.com



The route taken by the ice team has been carefully planned

A map is a way of picturing the spherical Earth on a flat piece of paper. Over small local areas, the surface of the Earth is very similar to a flat sheet, and so it is easy to accurately draw the geographical features of that area onto a piece of paper, keeping distances accurate.

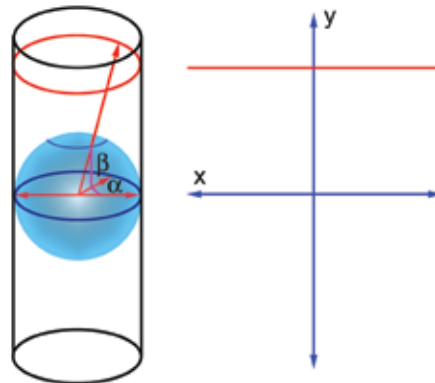
But what if you are interested in mapping a much larger area – the ice team are covering a very long distance – or perhaps even the whole globe itself? If you imagine wrapping a sheet of paper around an orange, at some point you will have to fold or scrunch up the paper in order to make it fit around the spherical orange. Any two-dimensional map of the whole globe is not going to accurately represent the spherical Earth. So just how do you map the sphere onto the 2D page?



Image © Alabama Maps <http://alabamamaps.ua.edu>

The Mercator projection

Map-makers use what is called a *projection* to depict a three-dimensional object (such as the globe of the Earth) in two dimensions. One of the most famous maps of the world is called Mercator’s projection. Invented in the 16th century by the Flemish cartographer Gerardus Mercator, this map is a projection of the globe onto a cylinder wrapped around the equator, which is then cut along one side to create a two-dimensional map.



The Mercator projection projects the Earth on a cylinder and then cuts it open

The simplest way to project the surface of the sphere onto the cylinder is to draw a line from the centre of the Earth through each point on the surface, and mark where that line intersects the cylinder. Then each point on the Earth’s surface (given by its longitude α and its latitude β) is mapped to a point (x,y) on the 2D map with coordinates

$$x = \alpha$$

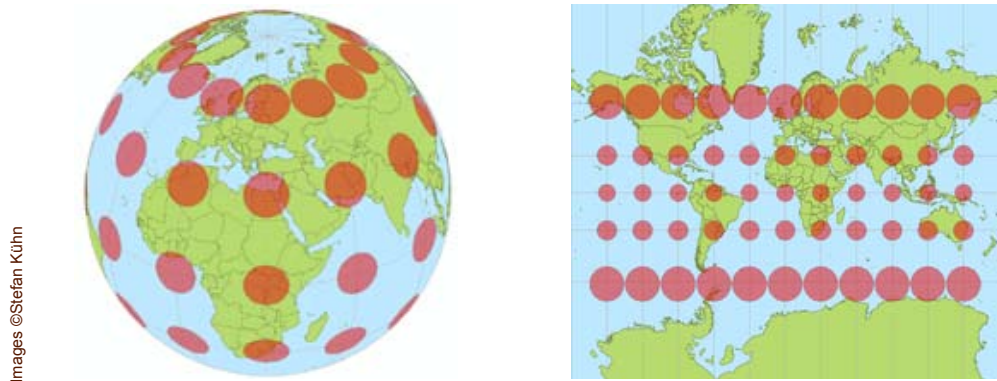
$$y = R \tan (\beta)$$

Here R is some constant which scales the map so it can fit on your page. If we assume $R=1$, then London, with longitude and latitude approximately $(0,51)$, maps to the point with coordinates $(0, 1.23)$, St John’s on the coast of Newfoundland, Canada, with longitude and latitude approximately $(52, 47)$, maps to $(52, 1.07)$. But as you get closer and closer to the North and South Poles, the distances get stretched, until the Poles actually disappear off the map altogether ($\tan (90)$ and $\tan (-90)$ are not defined).

Navigation Toolkit – Background article

Mercator modified this projection, by taking the y coordinate in his map to be $y=R \ln(\tan(\beta/2+45))$

This has the effect of reducing the distortion of the distances close to the Poles, but distance and area far from the equator are still exaggerated (eg. Greenland appears to be larger than Africa, when in reality Africa is over 14 times bigger).



Images ©Stefan Kühn

The distortion of the equally sized circles shows how the Mercator Projection distorts area

Although Mercator’s projection comes with distortion, it does have one important feature that means it is still in use for navigation today – it maps lines of constant bearing on the globe to straight lines on the map. This makes Mercator’s map very useful for sailors and nautical navigation – to sail from London to Newfoundland, they measure the bearing by drawing a straight line connecting the two places on Mercator’s map and set sail in the direction indicated by the orientation of the straight line.

But there are also other projections. The Gall-Peters projection, for example, preserves area, but greatly distorts shapes. Some people prefer this map for political reasons: in Mercator’s projection Europe appears much larger and almost at the centre of the map, while the Gall-Peters projection depicts relative sizes accurately.



Image ©NASA.

The Gall-Peters projection

The important point to realise is that no one projection is correct

In fact there are many projections used to map the world. The one we see in most atlases and world maps is the *Winkel tripel* projection – used by National Geographic for world maps, in the Times Atlas, and many school textbooks. This projection is more complex – it averages the coordinates from two other projections, the *Aitoff* projection and the *equirectangular* projection. The Winkel tripel projection doesn't preserve area or distance, but gives a balance with no one type of distortion overweighing.

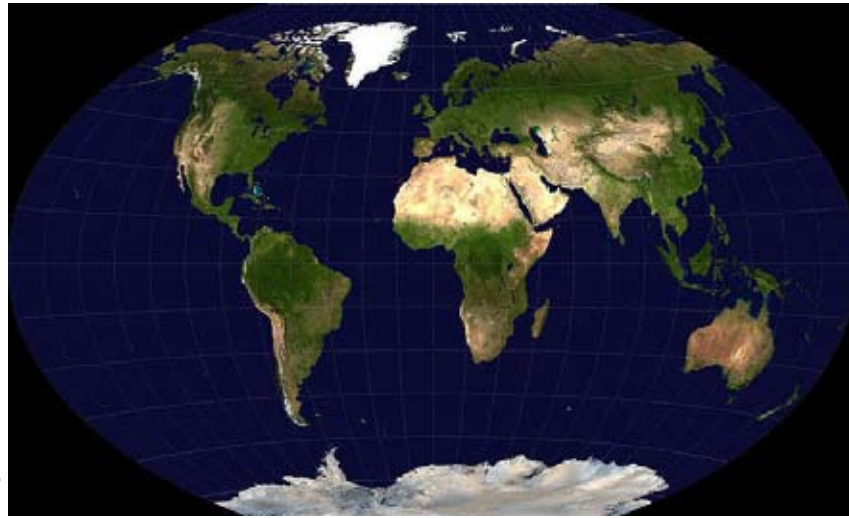


Image © NASA

The Winkel tripel projection

The important point to realise is that no one projection is correct – each has a distortion and you will never be able to create a map that accurately shows area, size, distance and direction for the whole globe at once. Instead you have to pick the best map for the job.

So what map should our intrepid explorers, Pen, Ann and Martin, use to plot their progress to the North Pole? The projections above are no use as they don't even show the Poles. One option might be to use a *transverse Mercator* projection where the cylinder wraps around the Earth on a line of longitude running through the Poles rather than around the equator.

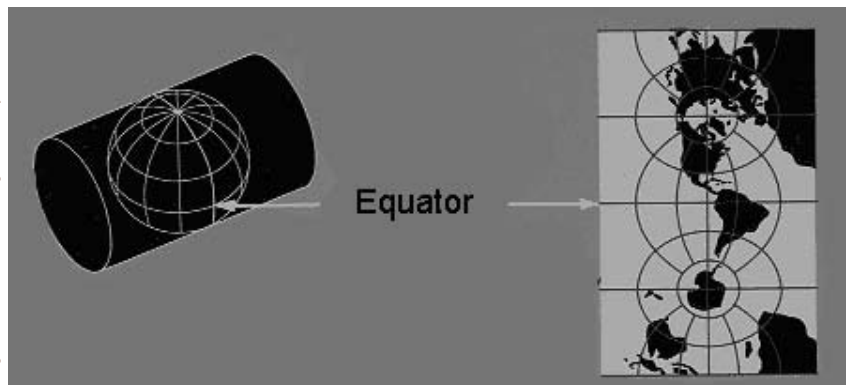


Image © United States Geological Survey

The transverse Mercator projection

Navigation Toolkit – Background article

Imagine balancing a stiff sheet of paper on top of the world

But perhaps the best choice would be a stereographic projection, like that used to produce the map showing the expedition route. This is a fairly simple projection: imagine balancing a stiff sheet of paper on top of the world, touching at the North Pole and projecting points from the South Pole onto the paper.

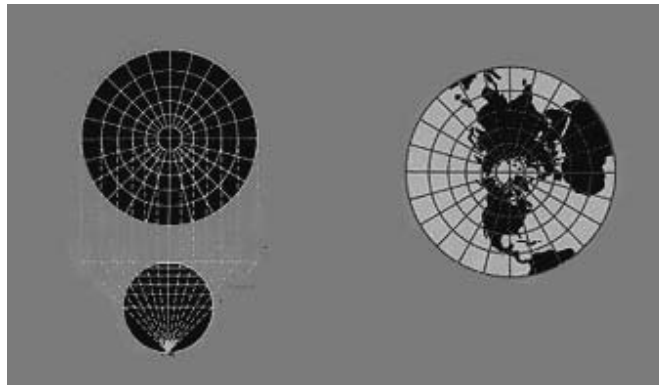
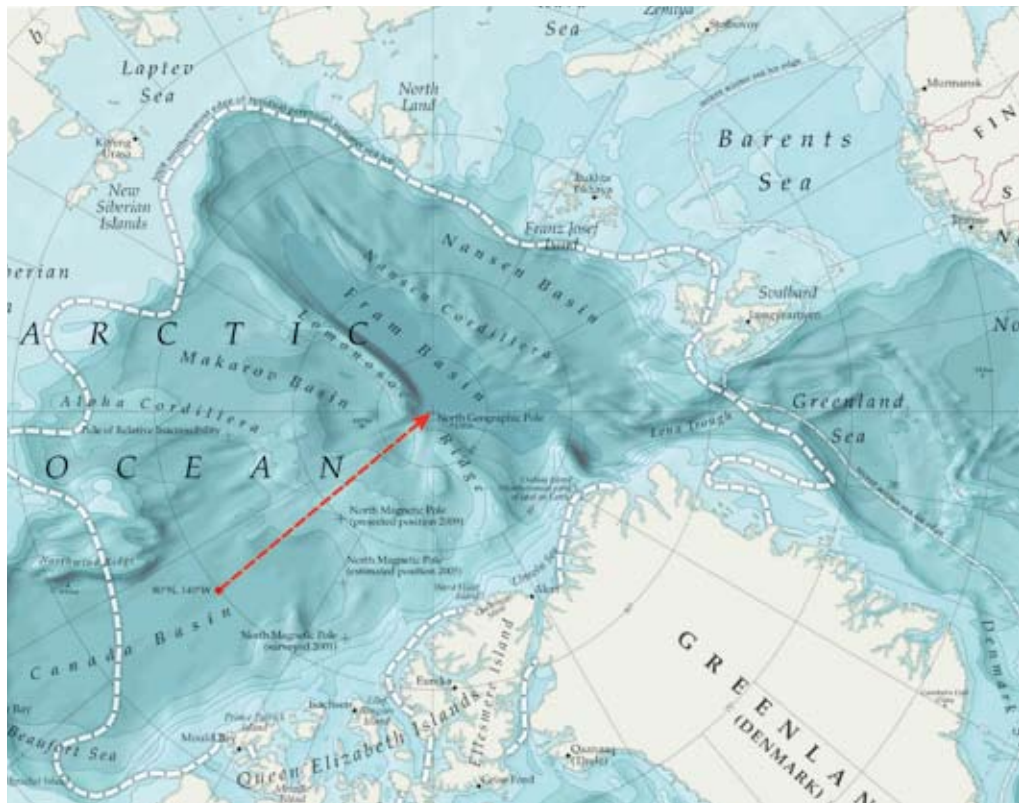


Image © United States Geological Survey.

The stereographic projection

This projection preserves angles. We can see the lines of latitude are concentric circles around the Pole in the centre of the picture and the lines of longitude are straight lines radiating out from the North Pole. Here is the map showing the explorers' intended route:



The map shows the intended route of the ice team

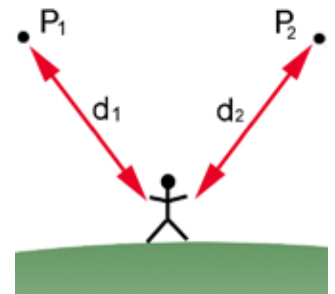
“Without GPS technology, the ice team would have to navigate by the stars!”

Simon Harris-Ward
Operations
Director

But where am I?

It's all very well having the right map, but if you don't know where you're standing, how can you know where you're going? There are few landmarks on the Arctic to navigate by – one iceberg looks much like another – and even the stars may not be visible in bad weather. Luckily, the ice team are equipped with GPS (the global positioning system). The GPS satellites that orbit the Earth continually transmit messages containing the exact time of transmission, and the exact location of the satellite at the time of transmission. When the GPS receiver receives these messages, it uses the time delay to work out exactly how far away it is from the satellite.

In the plane two points of reference are sufficient to work out your location: if you know that you are at distance d_1 from point P_1 and at distance d_2 from point P_2 , then you know that you are somewhere on the circle around P_1 with radius d_1 and somewhere on the circle around P_2 with radius d_2 . These two circles meet in at most two points, so you know that you are at one of these two points, and some additional information will help you to work out exactly which of the two it is.



In three dimensions a similar rule applies, although here you need three points of reference. If you know that you are at distance d_1 from point P_1 , distance d_2 from point P_2 and distance d_3 from point P_3 , then you know that you must be on the sphere around P_1 with radius d_1 , and on the sphere around P_2 with radius d_2 , and on the sphere around P_3 with radius d_3 . In general, two intersecting spheres intersect in a circle. A third sphere intersects that circle in at most two points. So again, with a little additional information, you can work out where you are once you know your distance from three points.

And this is exactly how GPS works: the GPS receiver uses the signals from at least three satellites to work out its distance from each, and then to work out its exact location by calculating the intersections of spheres. In fact, to minimise error and to gain additional information, GPS receivers use four satellites.

Further reading

- Thomas Harriott: a lost pioneer
<http://plus.maths.org/issue50/features/faherty/>
- Time and motion
<http://plus.maths.org/issue7/features/greatcircles/>
- The fundamentals of mapping
http://www.icsm.gov.au/mapping/about_projections.html
- Map projections
<http://erg.usgs.gov/isb/pubs/MapProjections/projections.html>
- Spherical projections
<http://local.wasp.uwa.edu.au/~pbourke/geometry/spherical>



photograph © Martin Harlley www.martinharlley.com

Ann checking her GPS position before setting off in the morning

MAPPING THE ARCTIC WITH MATHEMATICS

The aim of the Catlin Arctic Survey is to gather information to better understand the future of the Arctic sea ice cover. The route taken by the ice team has been carefully planned with the help of scientists using a detailed map of the Arctic – but how do you make such a map?

Question 1: Would you take the map below on a trip to the Arctic? What do you notice about the relative sizes of countries on this map?

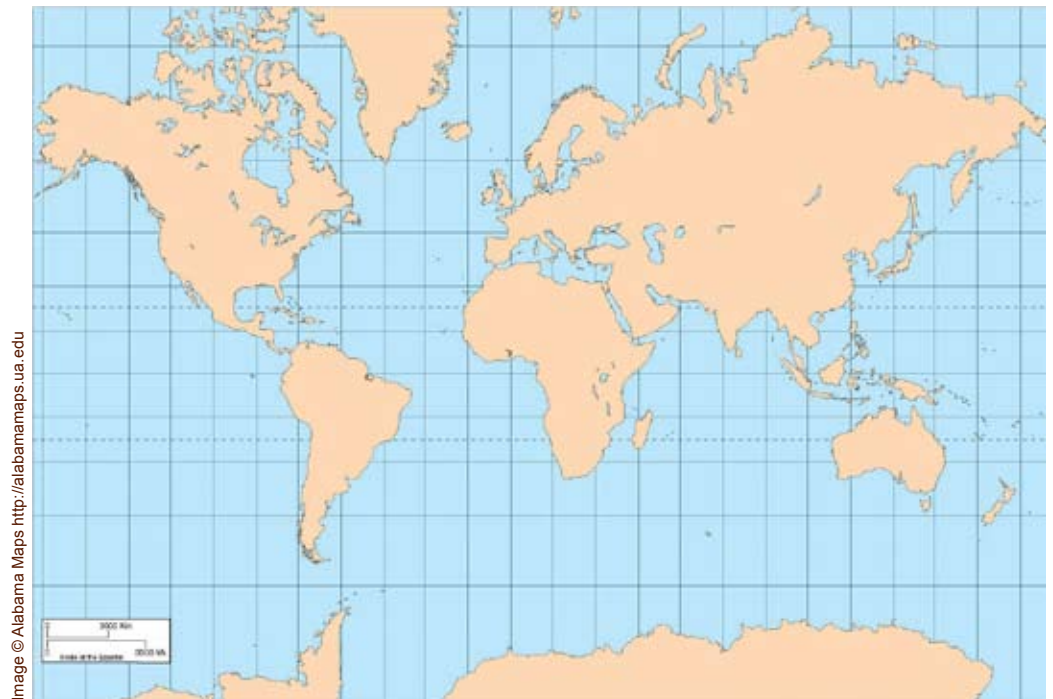


Image © Alabama Maps <http://alabamamaps.ua.edu>

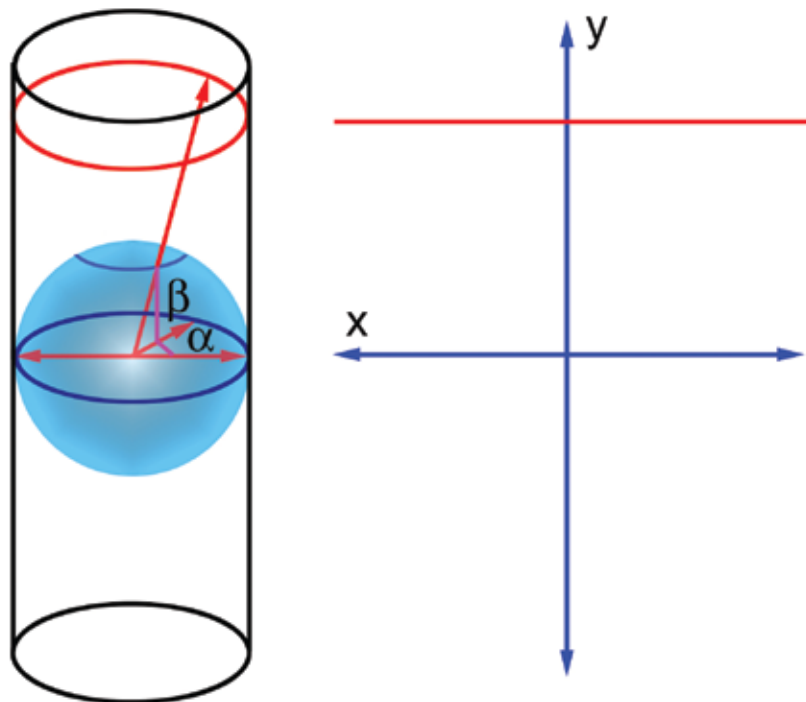
Any map is a two-dimensional representation of the Earth, which is a sphere. Any such projection comes with some degree of distortion. On this projection, called the *Mercator projection*, area is exaggerated as you move towards the Poles, and the Poles themselves are missing.

Navigation Toolkit – Mapping worksheet

“Good navigation tools are essential – our life depends on them!”

Pen Hadow
Expedition Leader

Here is a simplified description of how the Mercator projection works: place the spherical Earth into a cylinder of the same diameter, with the equator touching the cylinder. Now imagine placing a light at the centre of the Earth. Each point x on the surface of the Earth will have a ‘shadow’ y on the cylinder – y is the point where the straight line that connects x and the centre of the Earth hits the cylinder. In this way you can *project* the Earth onto the cylinder. Now cut the cylinder open (you can choose along which line to do this, but usually it is cut along the 180 degree meridian) – that’s the Mercator projection!

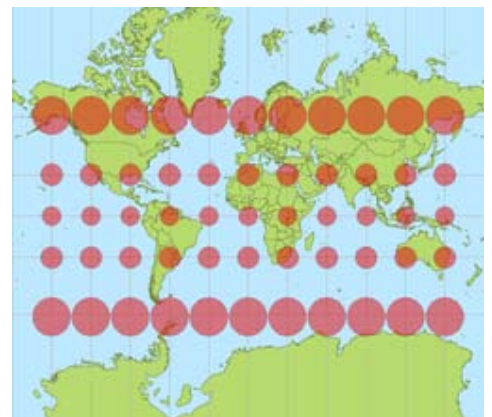


The mercator projection projects the Earth on a cylinder and then cuts it open

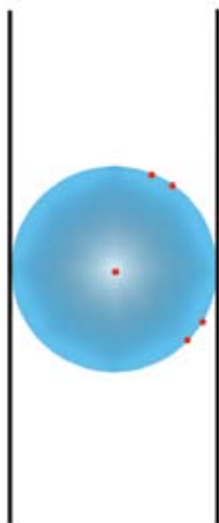
This is how discs of equal area are distorted:



Images ©Stefan Kühn

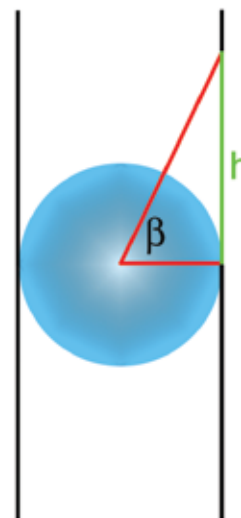


Navigation Toolkit – Mapping worksheet



Question 2: The diagram to the left shows a 2-dimensional representation of the projection set-up, viewed from the side. Find the projections of the points marked in red. What happens to the Geographic North Pole? Does this diagram explain the distortion? How tall a cylinder do you need to project the entire globe?

Question 3: In the diagram to the right the angle β describes the latitude of the point that has been marked. Assume that the radius of the Earth is 1 unit. Express the height h in terms of trigonometric functions of the unit circle.



Each point on the globe is given by its longitude α and latitude β . These are angles measured in degrees. The projection above maps each point to the point on the map with co-ordinates

$$\begin{aligned}x &= \alpha \\y &= R \tan(\beta).\end{aligned}$$

The number R is simply a scaling factor: if you want a very large map, you make R very large, and if you want a very small map, you make it small.

Question 4: Cairo has a latitude of approximately 30 degrees, London has a latitude of approximately 51 degrees, and Spitzbergen has a latitude of approximately 80 degrees. Setting $R=1$, use the formula above to work out their y -coordinate on a map. What do you notice? Can we use this projection to map the Poles?

Question 5: Sketch the graph of the function $y=\tan(\beta)$ for β ranging from -90 degrees to 90 degrees. How does this graph explain the distortion on the map?

Question 6: So the Mercator projection is useless for anyone wanting to explore the Arctic. Can you come up with another projection which represents the Arctic more accurately?

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Question 1: Would you take the map below on a trip to the Arctic? What do you notice about the relative sizes of countries on this map?

*** Answer:** No. The North Pole is not actually on this map at all and the area of countries is greatly exaggerated as you move towards the Poles. For example, Greenland appears bigger than Africa, but in reality its area is actually over 14 times smaller than that of Africa.



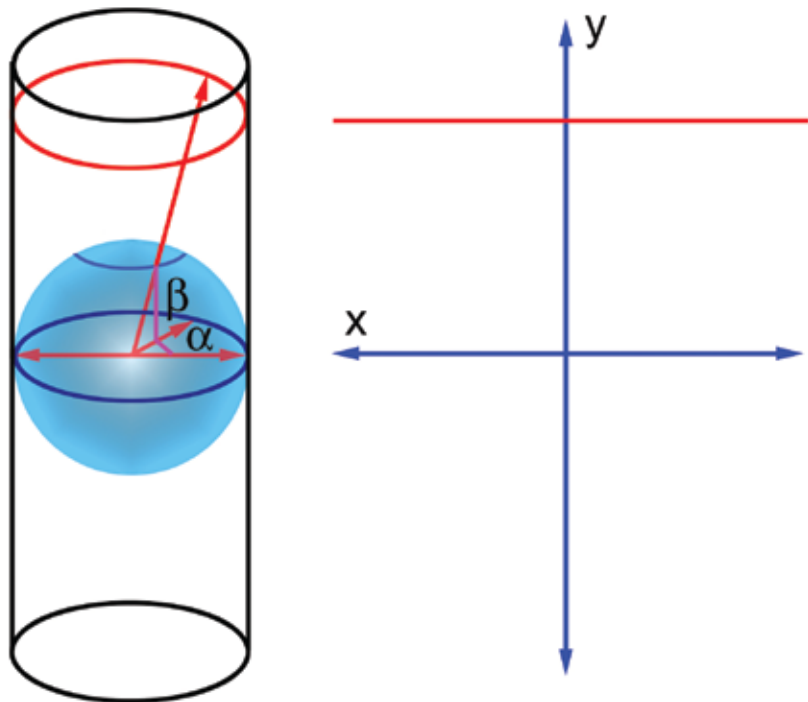
Image © Alabama Maps <http://alabamamaps.ua.edu>

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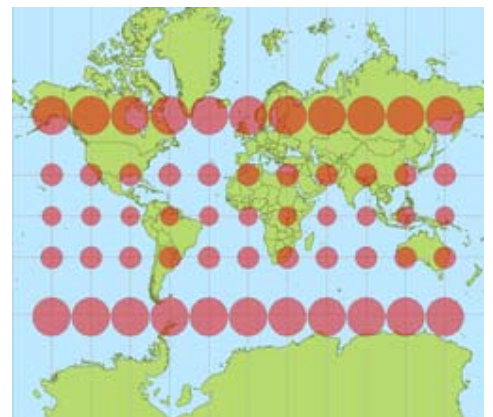


The mercator projection projects the Earth on a cylinder and then cuts it open

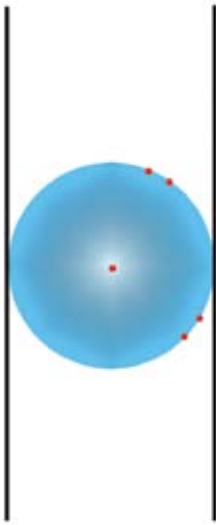
This is how discs of equal area are distorted:



Images ©Stefan Kühn

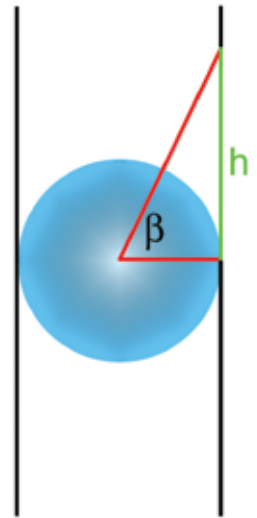


Navigation Toolkit – Mapping worksheet guidance with answers



Question 2: The diagram to the left shows a 2-dimensional representation of the projection set-up, viewed from the side. Find the projections of the points marked in red. What happens to the Geographic North Pole? Does this diagram explain the distortion? How tall a cylinder do you need to project the entire globe?

* **Answer:** *The North Pole has no projection because the straight line connecting the centre of the Earth and the North Pole never hits the cylinder at all. There is distortion in the map because the projection amplifies distances between points near the Poles. You'd need an infinitely tall cylinder to project the entire globe. So to construct a useful map, you need to decide on two 'cut-off latitudes': points on the Earth which lie further North or South than these latitudes will not be represented on the map.*



Question 3: In the diagram to the right the angle β describes the latitude of the point that has been marked. Assume that the radius of the Earth is 1 unit. Express the height h in terms of trigonometric functions of the unit circle.

* **Answer:** $h = \tan(\beta)$.

Navigation Toolkit – Mapping worksheet guidance with answers

Each point on the globe is given by its longitude α and latitude β . These are angles measured in degrees. The projection above maps each point to the point on the map with co-ordinates

$$\begin{aligned}x &= \alpha \\y &= R \tan(\beta).\end{aligned}$$

The number R is simply a scaling factor: if you want a very large map, you make R very large, and if you want a very small map, you make it small.

Question 4: Cairo has a latitude of approximately 30 degrees, London has a latitude of approximately 51 degrees, and Spitzbergen has a latitude of approximately 80 degrees. Setting $R=1$, use the formula above to work out their y-coordinate on a map. What do you notice? Can we use this projection to map the Poles?

***** **Answer:** Cairo has y-coordinate 0.577, London has y-coordinate 1.234 and Spitzbergen has y-coordinate 5.671. The y-coordinates become larger rapidly as you go North. The y-coordinates of the Poles are not defined.

Question 5: Sketch the graph of the function $y=\tan(\beta)$ for β ranging from -90 degrees to 90 degrees. How does this graph explain the distortion on the map?

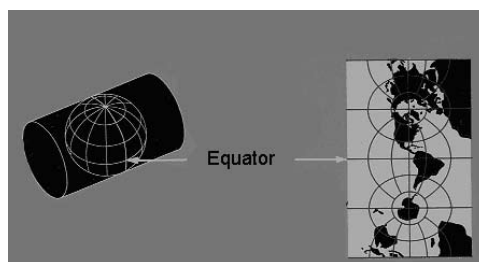
***** **Answer:** The distortion is reflected by the fact that the slope of the graph becomes larger and larger as you approach -90 degrees and 90 degrees.

Navigation Toolkit – Mapping worksheet guidance with answers

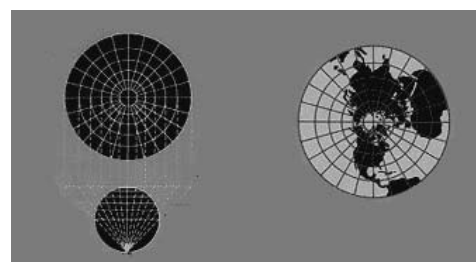
Question 6: So the Mercator projection is useless for anyone wanting to explore the Arctic. Can you come up with another projection which represents the Arctic more accurately?

*** Answer:** This is an open-ended question which is discussed in the accompanying article, but some possible projections include:

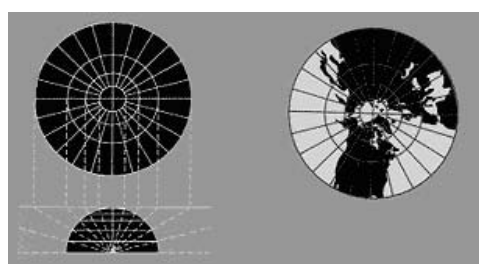
*** Transverse Mercator projection, which simply rotates the sphere so that the North and South Poles touch the cylinder.**



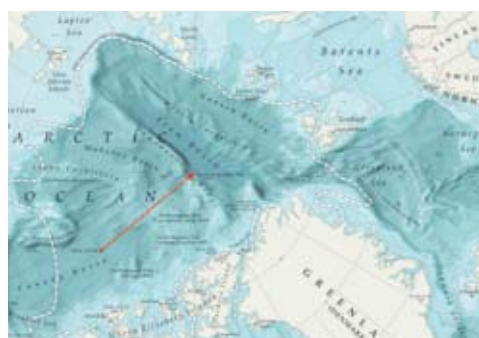
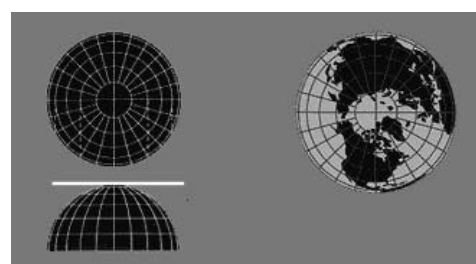
*** Stereographic projection, projecting a hemisphere onto a flat plane from a point on the Earth. For a map of the Arctic one would use the South Pole as the point to project from.**



*** Gnomonic projection projecting a hemisphere onto a flat plane from the centre of the Earth.**



*** Orthographic projection, projecting a hemisphere onto a flat plane from a point at infinity.**



*** The map showing the intended route of the ice team.**

FINDING YOUR WAY AROUND THE ARCTIC USING GPS

The GPS satellites that orbit the Earth continually transmit messages containing the exact time of transmission, and the exact location of the satellite at the time of transmission. When a GPS receiver receives these messages, it uses the time delay to work out exactly how far away it is from the satellite.



photograph © Martin Hartley www.martinhartley.com

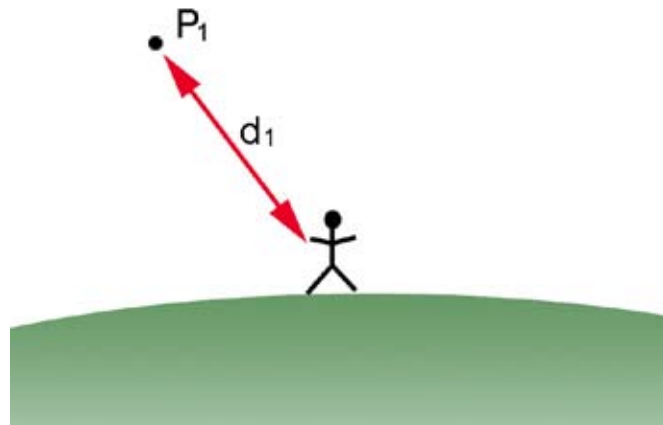
Ann checking her GPS position before setting off in the morning

Question 1: The signal transmitted by a satellite travels at the speed of light, which is approximately 300,000 km/s. The ice team's GPS receiver receives a signal from a satellite and calculates that the time the signal took to reach the receiver was 0.07 seconds. How far is the receiver from the satellite?

“Without GPS technology, the ice team would have to navigate by the stars!”

Simon Harris-Ward
Operations
Director

Question 2: Suppose the receiver has calculated that it is a distance d_1 away from a particular satellite (for simplicity, imagine that the satellite is stationary at point P_1). In the plane containing the satellite and the receiver, what kind of shape is formed by all the points that lie at distance d_1 from the satellite?



A two-dimensional view

Question 3: Is one satellite enough to calculate the explorers' exact location?

Question 4a: In a two-dimensional coordinate system mark the points P_1 and P_2 with coordinates $(0,20)$ and $(30,20)$ respectively. These denote the locations of two satellites that lie in a plane with the explorers (one unit in the coordinate system corresponds to 1000 km in our two-dimensional world, ground level lies on the x-axis). The receiver has calculated that the explorers are exactly 25 units (25,000km) away from each of the two satellites. Can you find the exact location of the explorers by drawing circles centred on the two satellites?

Question 4b (harder): In a two-dimensional coordinate system mark the P_1 and P_2 with coordinates $(0,20)$ and $(30,20)$ respectively. These denote the locations of two satellites that lie in a plane with the explorers (one unit in the coordinate system corresponds to 1000 km in our two-dimensional world, ground level lies on the x-axis). The receiver has calculated that the explorers are exactly 25 units (25,000km) away from each of the two satellites. Can you calculate the exact location of the explorers by using the equations of the circles centred on the two satellites?

Navigation Toolkit – GPS worksheet

Question 5: Now consider the more general case, when the satellites and the explorers do not all lie in a plane. In three dimensions, what kind of shape is formed by all the points at distance d from the satellite at point P ? Are two satellites enough to calculate the exact location of the explorers in three dimensions? How do questions 2 and 3 generalise to three dimensions?

Question 6a (challenging): Write down the equations of the sphere centred at the point $(0,15,0)$ with radius 25, and the sphere centred at $(0,-15,0)$ with radius 25. Observing the symmetry, what can you say about the circle of intersection?

Question 6b (challenging): What are the centre and radius of the circle C of intersection?

Question 6c (challenging): Observing the relative positions of the circle C of intersection and the sphere centred at the point $(0,0,40)$ with radius 20, find the point(s) where the sphere and the circle meet.

Conclusion: GPS systems are based on this process, which is called *trilateration*. However, to minimise errors and to gain extra information, real GPS systems calculate the distance between the receiver and four satellites.

FINDING YOUR WAY AROUND THE ARCTIC USING GPS

The GPS satellites that orbit the Earth continually transmit messages containing the exact time of transmission, and the exact location of the satellite at the time of transmission. When a GPS receiver receives these messages, it uses the time delay to work out exactly how far away it is from the satellite.



photograph © Martin Hartley www.martinhartley.com

Ann checking her GPS position before setting off in the morning

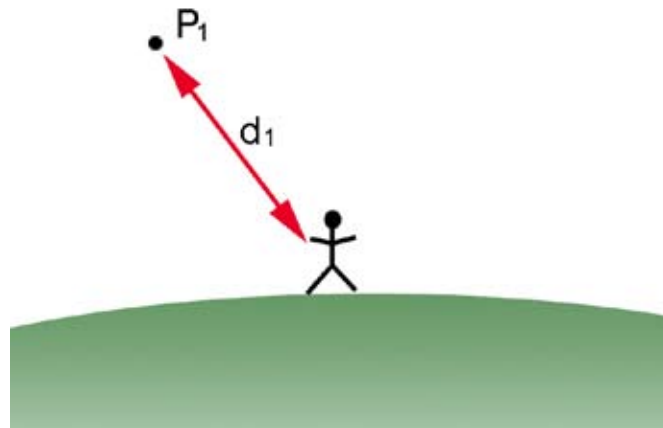
Question 1: The signal transmitted by a satellite travels at the speed of light, which is approximately 300,000 km/s. The ice team's GPS receiver receives a signal from a satellite and calculates that the time the signal took to reach the receiver was 0.07 seconds. How far is the receiver from the satellite?

*** Answer:** Write d for the distance between satellite and receiver in kilometres. Then $d/300,000 = 0.07$, so $d = 0.07 \times 300,000 = 21,000$, so the satellite is 21,000 km away from Earth.

“Without GPS technology, the ice team would have to navigate by the stars!”

Simon Harris-Ward
Operations
Director

Question 2: Suppose the receiver has calculated that it is a distance d_1 away from a particular satellite (for simplicity, imagine that the satellite is stationary at point P_1). In the plane containing the satellite and the receiver, what kind of shape is formed by all the points that lie at distance d_1 from the satellite?



A two-dimensional view



Answer: A circle with radius d and centre P_1 .

Question 3: Is one satellite enough to calculate the explorers' exact location?



Answer: No, if the satellite is not directly overhead the circle of points at distance d_1 would intersect the Earth's surface at two points, and either of those could be the location of the explorers. It would be possible to determine their exact location if the satellite is directly overhead, but there is no way to know if this is the case.

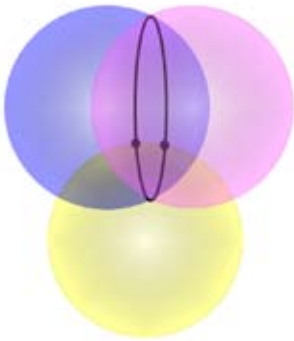
Question 4a: In a two-dimensional coordinate system mark the points P_1 and P_2 with coordinates $(0,20)$ and $(30,20)$ respectively. These denote the locations of two satellites that lie in a plane with the explorers (one unit in the coordinate system corresponds to 1000 km in our two-dimensional world, ground level lies on the x-axis). The receiver has calculated that the explorers are exactly 25 units (25,000km) away from each of the two satellites. Can you find the exact location of the explorers by drawing circles centred on the two satellites?

Question 4b (harder): In a two-dimensional coordinate system mark the P_1 and P_2 with coordinates $(0,20)$ and $(30,20)$ respectively. These denote the locations of two satellites that lie in a plane with the explorers (one unit in the coordinate system corresponds to 1000 km in our two-dimensional world, ground level lies on the x-axis). The receiver has calculated that the explorers are exactly 25 units (25,000km) away from each of the two satellites. Can you calculate the exact location of the explorers by using the equations of the circles centred on the two satellites?

*** Answer:** Using a compass to draw two circles around the two points gives coordinates $(15,0)$ and $(15,40)$, hence the explorers are at $(15,0)$ as they are at ground level. Same result if you use the circle equations $x^2+(y-20)^2=25^2$ and $(x-30)^2+(y-20)^2=25^2$.

When the satellites and explorers are all in a plane, the data from two satellites is enough to calculate the exact position of the explorers. Any two circles in the plane meet in one point, two points or not at all. Since the explorers are on each of the circles centred on the satellites, the two circles must meet. If they meet at just one point we are done. If they meet at two points, then we use the fact that we know which side of the satellites the explorers are (they are on Earth) to work out which of the two intersection points gives the location.

Navigation Toolkit – GPS worksheet guidance with answers



Two intersecting spheres generally meet in a circle and a third meets this circle in at most two points.

Question 5: Now consider the more general case, when the satellites and the explorers do not all lie in a plane. In three dimensions, what kind of shape is formed by all the points at distance d from the satellite at point P ? Are two satellites enough to calculate the exact location of the explorers in three dimensions? How do questions 2 and 3 generalise to three dimensions?

*** Answer:** In three dimensions, the points at distance d from a given point P lie on a sphere with centre P and radius d . The points that lie at distance d_1 from point P_1 and at distance d_2 from P_2 lie on a circle C formed by the intersection of the two spheres centred at P_1 and P_2 with radii d_1 and d_2 respectively. So we need at least one other satellite to determine the explorers' position. One other satellite is sufficient because a third sphere will intersect the circle C in at most two points. Additional information will then allow you to work out which of these is the location of the explorers.

Question 6a (challenging): Write down the equations of the sphere centred at the point $(0, 15, 0)$ with radius 25, and the sphere centred at $(0, -15, 0)$ with radius 25. Observing the symmetry, what can you say about the circle of intersection?

*** Answer:** Sphere 1: $x^2 + (y - 15)^2 + z^2 = 25^2$
Sphere 2: $x^2 + (y + 15)^2 + z^2 = 25^2$.
As the centres lie 15 units either side of the origin along the y axis, and since the spheres have the same radius, the circle of intersection must lie in the (x, z) plane and be centred at $(0, 0, 0)$.

Question 6b (challenging): What are the centre and radius of the circle C of intersection?

*** Answer:** It's the circle centred at $(0, 0, 0)$ with radius 20

Question 6c (challenging): Observing the relative positions of the circle C of intersection and the sphere centred at the point $(0, 0, 40)$ with radius 20, find the point(s) where the sphere and the circle meet.

*** Answer:** The centres of the circle and the sphere both lie on the z -axis and are 40 units apart. Since the radii of the sphere and the circle are both 20, there is only one point of intersection, namely $(0, 0, 20)$.

Conclusion: GPS systems are based on this process, which is called *trilateration*. However, to minimise errors and to gain extra information, real GPS systems calculate the distance between the receiver and four satellites.

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